(12) 33 p.

(11) Oct 73

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Strongly regular graphs,

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Oct 73

0. Introduction

The theory of strongly regular graphs was introduced by Bose [7]
in 1963, in connection with partial geometries and 2 class association
schemes. One year later, Higman [34] initiated the study of the rank
3 permutation groups using the strongly regular graphs. Both combinatorial
and groupal aspects have been developed in the last years. Moreover,
the interest for strongly regular graphs has been stimulated by the
discovery of new simple groups. In this paper we have tried to
summarize the main results on this subject, and to include some new ones.
We have also included an extensive bibliography on s.r. graphs. is also included.

Throughout this paper we use the notations of Seidel [65] which
arise from the study of equiangular lines (see VanLint and Seidel [80]).
These notations are well adapted to the concept of complementation, and
to the complementary graph; in the first section we mention the connection
with other notations used by Bose and Higman which arise respectively from
the n-class association schemes and from the centralizer ring.

We have included as an appendix a table of all the s.r. graphs
(to the best of our knowledge) related to classical groups, to sporadic
groups and arising from combinatorial constructions.

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267 360

1. Basic definitions and notations

The graphs considered here are undirected, without loops and multiple edges.

Let $\mathcal{G}$ be such a graph and let v be the number of vertices. $\mathcal{G}$ is said to be regular if each vertex is adjacent to the some number k of vertices; k is the valency of $\mathcal{G}$. $\mathcal{G}$ is said to be strong if:

(i) given any two adjacent vertices x,y, the sum of the number of vertices adjacent to both x and y and the number of vertices non adjacent to x and y is constant.

(ii) given any two non adjacent vertices, the same sum is constant.

$\mathcal{G}$ is strongly regular if it is strong and regular; it implies that:

(i) the number of vertices adjacent to both endpoints of an edge is constant and equal to λ

(ii) the number of vertices adjacent to two non adjacent vertices is constant and equal to μ

We shall denote by $\Delta(p)$ (resp. $\Gamma(p)$) the set of vertices adjacent (resp. non adjacent) to a vertex p .

$\overline{\mathcal{G}}$ is the complement of $\mathcal{G}$ if the set of vertices of $\overline{\mathcal{G}}$ is the set of vertices of $\mathcal{G}$ and if two vertices in $\overline{\mathcal{G}}$ are adjacent if and only if they were not adjacent in $\mathcal{G}$. If $\mathcal{G}$ is regular, then $\overline{\mathcal{G}}$ is regular of valency $\ell = v - k - 1$. Moreover if $\mathcal{G}$ is strong, $\overline{\mathcal{G}}$ is strong.

Let $\mathcal{H}_1 \cup \mathcal{H}_2$ be a partition of $\mathcal{G}$. The graph $\mathcal{G}'$ obtained by complementation or switching with respect to $\{\mathcal{H}_1, \mathcal{H}_2\}$ is the

graph whose vertices are the vertices of $\mathcal{G}$; all the adjacencies in $\mathcal{H}_1$ and $\mathcal{H}_2$ are preserved, but a vertex of $\mathcal{H}_1$ is joined to a vertex of $\mathcal{H}_2$ if and only if they were not joined in $\mathcal{G}$. If $\mathcal{G}$ is strong, then $\mathcal{G}'$ is again a strong graph.

To any graph one may associate an <u>adjacency matrix</u> A. Let the vertices of $\mathcal{G}$ be labelled by $1,2,\ldots,v$. Now construct the $v \times v$ matrix A whose entries are defined in the following way:

$a_{ij} = \alpha$ if $i = j$

$a_{ij} = \beta$ if i and j are adjacent

$a_{ij} = \gamma$ is i and j are not adjacent.

If $\mathcal{G}$ is a strongly regular graph it can be easily shown that the matrix algebra generated by A, I and J (the matrix with "1" in each entry) is of dimension 3. A straightforward calculation gives

$$A^2 = [2(\alpha-\gamma) + (\lambda-\mu)(\beta-\gamma)]A$$
$$+[k(\beta-\gamma)^2 + \lambda(\gamma-\alpha)(\beta-\gamma) + \mu(\alpha-\beta)(\beta-\gamma) - (\alpha-\gamma)^2]I$$
$$+[k\gamma(2\beta-\gamma) + \ell\gamma^2 - \lambda\gamma(\beta-\gamma) + \mu\beta(\beta-\gamma) + \gamma^2] J$$

and

$$AJ = (\alpha + k\beta + \ell\gamma) J .$$

In particular if $\alpha = 0$, $\beta = -1$, $\gamma = 1$, A is the $(0,-1,1)$ adjacency matrix of $\mathcal{G}$ and it satisfies:

$$A^2 + 2(\lambda-\mu+1) A - (4k-2\lambda-2\mu-1) I = (v-4k+2\lambda+2\mu) J$$

$$AJ = (\ell-k)J$$

The eigenvalues of A are $\rho_0 = \ell-k$ and ρ_1, ρ_2 which are roots of the quadratic equation:

$$x^2 + 2(\lambda-\mu-1)x - (4k-2\lambda-2\mu-1) = 0 , \quad (\rho_1 > 0 > \rho_2)$$

The multiplicities m_1, m_2 of ρ_1, ρ_2 are respectively

$$[-\rho_0 - \rho_2(v-1)](\rho_1-\rho_2)^{-1} \quad \text{and} \quad [\rho_0 + \rho_1(v-1)](\rho_1-\rho_2)^{-1} \quad .$$

Looking at these multiplicities, one finds necessary conditions for the existence of a strongly regular graph; either

(I) $k = \ell$, $\mu = \lambda+1 = k/2$

or   (II) $d = (\lambda-\mu)^2 + 4(k-\mu)$ is a square and $\sqrt{d}$ divides $2k + (\lambda-\mu)(v-1)$. Moreover if v is even, $2\sqrt{d}$ does not divide this quantity; but if v is odd, it does.

In the second case, the eigenvalues ρ_1, ρ_2 are odd integers. In the first case $\rho_0 = 0$ $\rho_1 = -\rho_2 = \sqrt{v}$ and it is known that a necessary condition for the existence of such graphs is that $v = a^2 + b^2$, where a and b are integers of different parity. Graphs of this type have been constructed for all admissible $v = p^\alpha$, p prime, and for some other values.

Let us mention that the $(0,-1,1)$-adjacency matrix A of a strong graph satisfies the equation $(A-\rho_1 I)(A-\rho_2 I) = (v-1 + \rho_1\rho_2)J$. If $v-1 + \rho_1\rho_2 \neq 0$, Seidel has shown [65] that the graph is strongly regular.

Complementation with respect to $(\mathcal{H}_1, \mathcal{H}_2)$ transforms the matrix A to a matrix A' such that

$a'_{ij} = a_{ij}$ if both i and j are in $\mathcal{H}_1$ or $\mathcal{H}_2$

$a'_{ij} = a_{ij}$ if i and j are not in the same $\mathcal{H}_k$

The matrix A of the complementary graph $\mathcal{G}$ of $\mathcal{G}$ is equal to $-A$; we have

$$\bar{v} = v , \ \bar{k} = \ell , \ \bar{\ell} = k , \ \bar{\lambda} = \ell - k + \mu - 1, \ \bar{\mu} = \ell - k + \lambda + 1,$$
$$\rho_0 + \bar{\rho}_0 = \rho_1 + \bar{\rho}_2 = \rho_2 + \bar{\rho}_1 = 0$$

Other notations used

A) Higman [34] considers the (0,1) adjacency matrix of a graph defined by $\alpha = 0$, $\beta = 1$, $\gamma = 0$; this matrix satisfies the equation $A^2 - (\lambda-\mu)A - (k-\mu)I = \mu J$ and $AJ = kJ$. The eigenvalues s and t are related to ρ_1 and ρ_2 by $\rho_1 = -(2s + 1)$, $\rho_2 = -(2t + 1)$

B) With the terminology of the 2-class association schemes, see Bose [7], the vertices of the graph are the varieties; two varieties are first (resp. second) associate if the two corresponding vertices are adjacent (resp. non adjacent) the correspondence between the notations are :

$$k = n_1 \ , \ \ell = n_2 \ , \ \lambda = p_{11}^1 \quad \text{and} \quad \mu = p_{11}^2 \ .$$

2. Partial geometries

A partial geometry is a set of elements called points, together with a set of subsets called lines such that

   (i) every pair of points lie on at most one line

   (ii) every line contains K points, $k \geq 2$

   (iii) every point is on R lines $R \geq 2$

   (iv) given a pair of non incident point-line (p.L), there are exactly T lines on p incident with L, $K \geq T \geq 1$.

Any partial geometry yields a strongly regular graph $\mathscr{G}$ defined as follows: the vertices of $\mathscr{G}$ are the points of the geometry; and two vertices are adjacent if and only if the two corresponding points are on a line; in other words $\mathscr{G}$ is the point-graph of the geometry The parameters of the graph may be computed from the parameters (R,K,T) of the partial geometry. We have:

$$v = KT^{-1}[(R-1)(K-1)+T]$$

$$k = R(K-1)$$

$$\ell = (K-1)(R-1)(K-T)T^{-1}$$

$$\lambda = K-2 + (R-1)(T-1)$$

$$\mu = RT$$

Obviously these numbers are integers, hence T divides $K(K-1)(R-1)$. Moreover the graph $\mathcal{G}$ is strongly regular; the necessary conditions for the existence of a s.r.graph imply that $T(R+K-T-1)$ divides $RK(K-1)(R-1)$.

A graph $\mathcal{G}$ having a set of parameters (v,k,ℓ,λ,μ) such that there exists integers (RKT) satisfying the above relations is called __pseudo-geometrizable__; it is __geometrizable__ if such a geometry exists.

A pseudo geometric graph does not always arise from a partial geometry; for instance there exists a graph with parameters $v = 16$, $k = 6$, $\lambda = 2$ which corresponds to $R = 2\ K = 4\ T = 1$ and this graph is not isomorphic to the point-graph of the unique partial geometry $(2,4,1)$.

A theorem of Bose [7] give a sufficient condition for a pseudo geometric graph to be geometrizable.

There exists in $\mathcal{G}$ a set Σ of cliques (complete subgraphs) such that 1) two adjacent vertices are in exactly one clique of Σ, 2) every vertex belongs to exactly R cliques of Σ, 3) K, the common size of all Σ-cliques, is greater than R. Another result of Bose gives a sufficient condition; a pseudo-geometric graph $\mathcal{G}$ is geometrizable if

$$K > \frac{1}{2} [R(R-1) + T(R+1)(R^2-2R + 2)]$$

Given any partial geometry with parameters (R, K, T), the dual, i.e. the geometry whose points and lines are the lines and the points of the first one, is again a partial geometry with parameters (K, R, T).

Those partial geometries have been investigated by Bose [7], Sims [73], Higman [11] and Ahrens and Szekeres [1].

3. t-Designs and Symmetrics 2-designs

A _t-design_ $S_\lambda(t,k,v)$ is a set of v _points_ with subsets of size k, called _blocks_, such that every t distinct points belong to exactly λ blocks; the number b of blocks is given by

$$b = \lambda \left(\begin{array}{c} v \\ t \end{array} \right) / \left(\begin{array}{c} k \\ t \end{array} \right) .$$

If $t = 2$, it may happen that the number of blocks equals the number of points. In this case $\lambda(v-1) = k(k-1)$. The simpliest examples are given by the projective spaces (the blocks are the hyperplanes). Let us mention that in a symmetric 2-design, two blocks intersect in 2 points and there are k blocks containing each point.

In some particular case it is possible to derive a symmetric design from a strongly regular graph.

A) if $\lambda = \mu$, take as blocks the sets $\Delta(p)$ for every vertex p of $\mathcal{J}$. This yields a $S_\lambda(2,k,v)$ symmetric design

B) if $\lambda = \mu-2$, take or blocks the sets $\{p\} \cup \Delta(p)$ for every vertex p of $\mathcal{J}$. One obtains the symmetric design $S_\mu(2,k+1,v)$

Let us notice that if $\lambda = \mu-2$, $\overline{\lambda} = \overline{\mu}$ in the complementary graph $\mathcal{J}$ and the symmetric design is the complement of the design

obtained from the complementary graph.

In case A, $\lambda = \mu$, the necessary conditions for the existence of a strongly regular graph reduce to:

$4(k-\lambda) = d$ is a square

$\sqrt{d}$ divides $2k$

Hence $k - \lambda = m^2$ and m divides k; this implies m divides λ. Thus for a given λ there are finitely many strongly regular graphs with $\lambda = \mu$.

These graphs, sometimes called (v,k,λ)-graphs, have been studied by Bose and Shrikande [10], Rudvallis [62] and Wallis [83]. Rudvallis [62] proved that the existence of a (v,k,λ) graph is equivalent to the existence of a symmetric $S_\lambda(2,k,v)$ admitting a polarity without absolute points. A necessary (but not sufficient) condition for the existence of such graphs (or designs) is that $(v,k,\lambda) = (s[(s+a)^2 - 1]/ a, s(s+a), sa)$ with a divisor of $s(s^2-1)$ and if a is even, s and $s(s^2-1)/a$ must be odd integers.

The connection between (v,k,λ) graphs and some symmetric designs has been studied by Hall, Lane and Wales [30] and Hubaut [49].

4. Equiangular lines

A set of v lines through one point in a r-dimensional euclidean space is called equiangular if the angle between every pair of lines is the same. An interesting problem is to determine the maximum number $v(r)$ of such lines in a r-dimensional space Seidel [67] proved that is is possible to derive from any symmetric $(0,-1,1)$ v x v matrix A a set of equiangular lines in a r-dimensional space. If ρ is the smallest eigenvalue (necessarily negative) with

multiplicity $v-r$, then the matrix $\frac{1}{\rho}$ $(A-\rho I)$ may be interpreted as
the grammian matrix of v vectors in a r-space; the angle and
between two lines satisfies $\cos\alpha=\frac{1}{\rho}$. In the case of strongly
regular and strong graphs, the multiplicity of the smallest eigenvalues
if often very large and therefore the number v of equiangular lines
is large with respect to r . Moreover there exists a connection
between these set of lines and a regular polyhedron, so that for
some sets of equiangular lines, it happens that the automorphism
group of the corresponding graph is very large. More details
may be found in Lemmens and Seidel [52]

5. Rank 3 graphs.

Let G be a transitive permutation group on a set Ω . If
G_p , the subgroup of G fixing $p\in\Omega$, has r orbits, then G is
said to be a rank r group. In the case where $r=3$ let the 3
orbits be $\{p\}$, $\Delta(p)$, $\Gamma(p)$. It is obvious that $q\in\Delta(p)\Rightarrow p\in\Delta(q)$
holds if and only if G is of even order. In this case it is possible
to derive from G a strongly regular graph $\mathcal{G}$ whose set of vertices
is Ω ; two vertices p and q are adjacent in $\mathcal{G}$ iff $p\in\Lambda(q)$.

Higman [34 to 42] has developed the theory of rank 3 groups;
they act as an automorphism group on $\mathcal{G}$, transitive on the vertices
and on the edges.

Infinite classes of strongly regular graphs arise in the study
of representations of classical groups, especially simple groups.
Most of them have rank 3 representations and are a normal subgroup
of $\text{Aut}(\mathcal{G})$. In some cases there are higher rank representations but
it may happen that they yield strongly regular graphs.

A result of Seitz [68] gives important information about the rank 3 representations of Chevalley groups. There exists for each Chevalley group $G(q)$ an integer N such that if $q > N$, the only rank 3 representations of $G(q)$ are representations on the cosets of a parabolic subgroup.

Such rank 3 representations do not occur for $G_2(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$.

Most of the representations have been discovered by geometrical means. We would mention the papers of Primrose [58] and Ray-Chaudhuri [50] for the orthogonal groups, Bose and Chakravarti [8] and Chakravarti [12,13] for unitary groups, Higman and McLaughlin [43] for symplictic and unitarian groups and a series of papers by Wan-Zhe Xian, Yang Ben-Fu, Dai-Zong Duo and Fen-Xuning [19,22,85,86, 87,88] on classical groups.

Exceptional representation of $PO_{2n+1}(3)$ have been discovered by Rudvallis [62]. Other exceptional graphs and designs related to $V_{2n} \cdot O_{2n}^{\pm}(2)$ have been constructed by Mann [54] and also, in connection with coding theory, by Delsarte and Goethals [20]. We would also quote a result of Taylor [76] about strong graphs with $PSU_n(q^2)$ as an automorphism group. For sporadic groups having rank 3 representations, we refer the reader to Tits [78].

6. Some results about rank 3 graphs

A. General results

Foulser [24] and Dornhoff [21] have determined the primitive rank 3 solvable groups G. The corresponding graphs have parameters $(v,k,\lambda) = (i)(n^2,\ g(n-1),\ (g-1)(g-2) + n-2)$, i.e. the graph is of type

$L_g(n)$ (K1) . The only possible values for (n,g) are $(3^2,4)$, (13,6) (17,6) (19,8) $(3^3,4)$ (29,6) (31,8) (47,24) or $(3^2,4)$ $(7^2,10)$

(ii) (64,27,10)

Moreover there exists two other classes when G is a subgroup of the affine group of the line or when G acts on a vector space V such that $V = V_1 \oplus V_2$, $V_1 \cup V_2$ being a set of imprimitivity for G_0

When the rank 3 group acts on an affine plane, Kallaher [51] and Liebler [53] proved that the plane is a translation plane.

If a rank 3 group possesses a normal regular subgroup, it is ismorphic to a subgroup of automorphisms of the affine space $AG(n,g)$ containing the translations.

B. Characterization by the constitions

Tsuzuku [79] has determined the primitive rank 3 extensions of Sym (k) acting in its natural representation on k points. The following extensions occur: if $k > 1$

(i) $k = 2$ $v = 5$ and $G \simeq D_5$ dihedral of order 10

(ii) $k = 3$ $v = 10$ $\mathcal{I}$ is the Petersen graph $G \simeq$ Sym (5)

(iii) $k = 5$ $v = 16$ $\mathcal{I}$ is the Clebsch graph $G \simeq 2^4$. Sym(5)

(iv) $k = 7$ $v = 50$ $\mathcal{I}$ is the Hoffman-Singleton graph

and $G \simeq PSU_3^x(5^2)$

Iwasaki [50] proved that the same result holds for $G_0|_\Delta =$ Alt(k) in its natural representation. Montague [57] has shown the non existence of most extensions of 2-transitive groups on k points with $|\Delta| = k$, $|\Gamma| = \frac{k(k-1)}{2}$. There are $PSL_2(q)$, $PSU_3(q^2)$, R(q), $S_z(q)$ in their natural representation on $k = q + 1$, $q^3 + 1$, $q^3 + 1$ points. The only exceptional cases are

(i) $PSL_2(2) \simeq Sym(3)$: the extension is the Petersen graph.

(ii) $PSL_2(4) \simeq Alt(5)$ which extends in the Clebsch graph

(iii) $PSL_2(9)$ which gives $PSL_3(4)$ acting on the graph
(56,10,0)(Sl).

If the groups $G_{0|\Delta}$ and $G_{0|\Gamma}$ are 2-transitive on both Δ and Γ, the graph is isomorphic either to a pentagone , or to the union of two complete graphs of order $n = \dfrac{v}{2}$. In the last case the normal subgroup fixing each component must be 3-fold transitive on K_n .

Bannai [3] proved that if $G_{0|\Delta} \simeq PSL_n(2^f)$ in its natural representation, then $n = 2$ and $f = 1$ or 2 . These cases are covered by the result of Montague. In another paper Bannai [4] studied the case where $G_{0|\Delta}$ is 4-fold transitive; he showed that $|\Delta| = 5$, 7 and $G_{0|\Delta} \simeq Sym(5)$, $Sym(7)$ or $Alt(7)$

C. Characterization by subdegrees

In some case, the knowledge of the subdegrees k and ℓ, together with the rank 3 assumption is sufficient to classify the rank 3 graphs. Higman proved [39]

(i) $v = m^2$ $k = 2(m-1)$ $m \geq 2$, then $G \simeq Sym(m) \backslash Sym(2)$ and $\mathcal{G}$ is of type $L_2(m)$

(ii) $v = \binom{m}{2}$ $k = 2(m-2)$ $m \geq 5$ then G is a 4-fold transitive subgroup of $Sym(m)$ and $\mathcal{G}$ is of type $T(m)$, or

   a. $G \simeq P\Gamma L_2(8)$ and $\mathcal{G}$ is of type $T(g)$

   b. $\mu = 6$ $m = 9, 17, 27, 57$

   c. $\mu = 7$ $m = 51$

   d. $\mu = 8$ $m = 28, 36, 325, 903, 8128.$

The only known case is $G_2(2)$ on 36 points (3.9)

(iii) $v = \dfrac{Q_n Q_{n-1}}{Q_2}$ $k = q\, Q_2$ with $Q_n = \dfrac{q^n-1}{q-1}$

then G is a subgroup of $P\Gamma L_m(q)$ acting on the lines of $P_{m-1}(q)$ transitive of the 4 simplices or possibly $m = 4$ or 5 either $m = 2a + 1$ and $17 \geq m \geq 7$ with $\mu \neq (q+1)^2$. An analogous result have been proved by Enomoto [22]; If $v = m^2$ and $k = 3(m-1)$; then $\mu = 6$ unless $\mu = 4$, $m = 14$ or 352. Furthermore if one makes the rank 3 assumption then if $m > 23$, beside the two exceptional cases, G is the automorphism group of a graph of type $L_3(n)$ with $n = 2^f$.

Let us mention that in these results, the condition on the parameters are independent of the rank 3 assumption.

Another result of Higman [35] for graphs with $\lambda = 0$, $\mu = 1$ i.e. with $v = k^2 + 1$, is that k must be $2,3,7,57$. These graphs exist and are rank 3 for $k = 2,3,7$. Ashbacher [2] proved that if $k = 57$ there is no such rank 3 graph.

If $\lambda = 0$ and $\mu \neq 2,4,6$ there are at most finitely may such graphs for fixed μ (Biggs [6]).

Higman [41] also studied the pseudo geometric graphs with $T = 1$ and $v > 100$; he proved, among other results, the non existence of rank 3 graphs with parameters $(76,21,2)$ and $(96,20,4)$ and the uniqueness of the rank 3 graph $(64,18,2)$.

Wales [81] has shown that the knowledge of $G_{0|\Delta}$ and $G_{0|\Gamma}$ determines uniquely the graph $\mathcal{J}$.

D. Characterization of s.r. graphs by the eigenvalues

Seidel [66] has determined all s.r. graphs with smallest eigenvalue $\rho_2 = -3$. They are:

(i) $L_2(n)$

(ii) $T(n)$

(iii) The non geometric graph $v = 16$ $k = 6$ $\lambda = 2$

(iv) the three Chang graphs [14]

(v) the graph of Petersen, Clebsh and Schlafli.

It is interesting to note that the last three form a rank 3 tower. Sims [73] proved, using a result of Ray-Chaudhuri [59] on line graphs, that a rank 3 graph with smallest eigenvalue $\rho_1 = -3$ is of type i,ii or v . Moreover he conjectured that if there exist infinitely many rank 3 graphs with smallest eigenvalue ρ_2, then $\rho_2 = 2q + 1$ $(q = p^{\alpha})$ and with finitely many exceptions they are graphs of type (C.1 and C11) . Another formulation is that the parameter μ is bounded by a function of the smallest eigenvalue. This conjecture is proved using a result of Hofmann.

7. Rank 3 towers

Let $\mathcal{G}$ be a rank 3 graph and $\Delta(p)$ $\Gamma(p)$ the two non trivial orbits of G_p . It may happen that $G_p|_\Delta$ is also a rank 3 representation of same group. This process may occur several times and yield a so-called rank 3 tower. The groups involved in the known tower are generally sporadic groups and some other "exceptional" simple group. We shall give the parameters of the known towers. For the first four the reader should refer to Tits [76] for more explanation.

	v	k	ℓ	λ	μ	G	G_0
Higman-Sims	77	60	16	47	45	M_{22}	$2^4 A_6$
Tower	100	77	22	60	56	HS	M_{22}
McLaughlin	162	105	56	72	60	$PSU_4(3)$	$PSL_3(4)$
Tower	275	162	112	105	81	McL	$PSU_4(3)$
Suzuki	36	14	21	4	6	$G_2(2) \cong PSU_3(3^2)$	$PSL_3(2) \cong PSL_2(7)$
	100	36	63	14	12	HJ	$G_2(2)$
Tower	416	100	315	36	20	$G_2(4)$	HJ
	1782	416	1365	100	96	Suz	$G_2(4)$
	693	180	512	51	45	$PSU_6(2^2)$	$2^9 . PSU_4(2^2)$
	3510	693	2816	180	126	Fi_{22}	$PSU_6(2^2)$
Fisher	$31 \cdot 671$	3510	$28 \cdot 160$	693	351	Fi_{23}	Fi_{22}
Tower	$306 \cdot 936$	$31 \cdot 671$	275263	3510	3240	Fi_{24}	Fi_{23}
Conway	1408	840	567	488	520	$PSU_6(2^2)$	$PSU_4(3^2)$
Tower	2300	1408	891	840	896	$Co \cdot 2$	$2 \cdot PSU_6(2^2)$
Mathieu	1288	792	495	476	504	M_{24}	$2 \cdot M_{12}$
Tower	2048	1288	759	792	840	$2^{11} . M_{24}$	M_{24}
Orthogonal	2^{2n}	$2^{2n-1} - 2^{n-1}$	$2^{2n-1} \mp 2^{n-1} - 1$	$2^{2n-2} - 2^{n-1}$	$2^{2n-2} \mp 2^{n-1}$	$2^{2n} . O^{+}_{2n}(2)$	$O^{+}_{2n}(2)$
Towers	$2^{2n+1} \mp 2^{n} - 1$	2^{2n}	$2^{4} \mp 2^{n} - 2$	$2^{2n-1} \mp 2^{n-1}$	2^{2n-1}	$O^{+}_{2n+2}(2)$	$2^{2n} . O_{2n}(2)$

In the last case, it should be noticed that in the first graph the suborbit $\Gamma(p)$ is again a orthogonal tower.

8. Strongly regular graph related to Chevalley groups (*)

1. $PSL_n(q)(A_{n-1}(q))$ acting in the lines of $PG(n-1,q)$ $n \geq 4$ adjacent vertices = intersecting lines

2. $P\Omega_{2n+1}(q)(B_n(q))$ acting on the points of a quadric in $PG(2n,q)$ adjacent vertices = point on a generatrix.

3. $PSp_{2n}(q)(C_n(q))$ acting on the points of $PG(2n-1,q)$ with a symplictic polarity ; adjacent vertices = conjugate points

4$^+$. $P\Omega_{2n}^+(q)(D_n(q))$ acting on the points of an hyperbolic quadric of $PG(2n-1,q)$; adjacent vertices = points on a generatrix.

4$^-$. $P\Omega_{2n}^-(q)(^1D_n(q^2))$ acting on the points of an elliptic quadric of $PG(2n-1,q)$; adjacent vertices = points on a generatrix.

5. $PSU_n(q^2)(^1A_{n-1}(q^2)$ acting in the points of an hermitian variety of $PG(n-1,q^2)$; adjacent vertices = points on a generatrix.

6. $E_6(q)$ acting on the points of the 26-dimensional projective representation; adjacent vertices = points on a generatrix.

7. $P\Omega_{10}^+(q)$ acting on one family of isotropic 4-planes of an hyperbolic quadric in $PG(g,q)$; adjacent vertices = 4 planes intersecting along a 2-plane.

8. $PSU_5(q^2)$ acting on the lines of an hermitian variety of $PG(4,q^2)$; adjacent vertices = intersecting lines.

Note: the same representation of $PSU_4(q^2)$ on the lines of an hermitian variety is isomorphic to (4-) with $n = 3$.

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* The full automorphism group of the graph is usually the automorphism group of the defining space except sometimes, for small values of q and n

9.$^{\pm}$ $P\Omega^{\pm}_{2n}(2)$ acting on the points off a quadric in $PG(2n-1,2)$ adjacent vertices = points on a tangent.

10.$^{\pm}$ $P\Omega_{2n+1}(3)$ acting on the points inside (resp outside) a quadric of $PG(2n,3)$; adjacent vertices = points on a non intersecting line.

Other graphs related to classical group

11. The stablizer of a coline in $PG(n,q)$ $(n \geq 3)$ acting on the non intersecting lines; adjacent vertices = intersecting lines.

12.$^{\pm}$ $V_{2n}(q).0^{\pm}_{2n}(q)$ acting on the points of a euclidean space $(V_{2n}(q))$ with a quadratic form); adjacent vertices = points on an isotropic line.

13. The subgroup $x' = ax^2 + b$ of the affine line over $GF(q)$ with $q = 1(\bmod\ 4)$ acting on the points; adjacent vertices = points whose coordinates differ by a square

Note: $12,13^{\pm}$, 14 are rank 3 representations

14. $PSU_n(q^2)$ $(^1A_{n-1}(q^2))$ acting on the points off an hermitian variety of $PG(n-1,q^2)$; adjacent vertices = points on a tangent.

15. Automorphism group of $AG(nq)$ acting on the lines; adjacent vertices = intersecting lines (rank 4 repr.)

16. Automorphism group of $AG(3q^2)$ with a Baer subplane at infinity; adjacent vertices = points on an " isotropic" line (rank 3 representation)

17. Automorphism group of $AG(3,2^r)$ with a complete conic at infinity (union of a conic and its knot), acting on the points; adjacent vertices = points on an "isotropic line" (rank 4 representation (except for $r = 2$, rank 3))

18. Some group acting on the isotropic lines; adjacent vertices = intersecting lines (non transitive repr. except for $r = 2$ rank 4)

19. Subgroup of automorphism of $AG(2,q)$ preserving m isotropic directions acting on the points; adjacent vertices = points on an isotropic line.

20. Automorphism group of an hermitian parabola P in $AG(2,q^2)$ (q odd). If the equation of P is $x\bar{x} + 2(y + \bar{y}) = 0$, two vertices are adjacent iff the corresponding points of coordinates $(x_1 y_1)$ and $(x_2 y_2)$ satisfy $x_1\bar{x}_2 + y_1 + y_2 + \bar{y}_1 + \bar{y}_2$ is a square (resp. a non square) in $GF(q)$ when $-1 \notin GF^{x2}(q)$ (resp $-1 \in GF^{x2}(q)$)

S   **Strongly regular graphs related to sporadic groups**

1. $PSL_3(4)$ acting on an orbit of 56 complete conics of $PG(2,4)$; adjacent vertices = disjoint conics (see Gewirtz [25], Goethals and Seidel [26] and Montague [57]). Rank 3 representation of $PSL_3(4)$ over $Alt(6)$.

2. M_{22} acting on the 77 blocks of $S(3,6,22)$; adjacent vertices = disjoint blocks. Rank 3 representation of M_{22} over $2^4.Alt(6)$.

3. $PSU_3(5^2)$ acting oversubsets of autoconjugate triangles in $PG(2,5^2)$ with an hermitian conic. A simple construction is obtained in the following way. Let $p = A_7$, $\Delta(p)$ is the set of 7 subgroups of A_7 isomorphic to A_6 (fixing one-letter) $\Gamma(p)$ is the set of 7×6 subgroups of each A_6 isomorphic to A_5 but transitive on the 6 letters. Two points of Γ are joined if the two A_5 intersect in D_5. See Hoffman and Singleton [48], Benson and Losey [5] and Schult [64] Rank 3 representation of $PSU_3(5^2)$ over $Alt(7)$.

4. $PSL_3(4)$ acting on the 105 flags of $PG(2,4)$. Two vertices are adjacent if the corresponding flags have distinct centers and axis

and if the center of one belongs to the axis of the other
(see also Seidel [67] and Goethals and Seidel [26]) Rank 6
representation.

5. $PSL_3(4)$ acting on an orbit of 120 Baer subplanes of $PG(2,4)$;
    adjacent vertices = planes intersecting in a single point
    (see Goethals and Seidel [26]) Rank 5 representation.

6. M_{23} acting on the 253 blocks of $S(4,7,23)$; adjacent vertices =
    blocks intersecting in a single point Rank 3 representation of
    M_{23} over $2^4.Alt(7)$.

7. M_{22} acting on the 176 blocks of M_{23} avoiding one point;
    adjacent vertices = blocks intersecting in a single point [26]
    Rank 3 representation of M_{22} over $Alt(7)$.

8. $PSU_3(5^2)$ acting on the 175 edges of the Hofman-Singleton graph
    (S3). Another description may be given using graph 7; in this
    graph, given a point p, take the subgraph $\Delta(p) \cup \Gamma(p)$ and
    switch with respect to (Δ,Γ). Rank 4 representation of $PSU_3(5^2)$
    over $2.Alt(6)$.

9. Rank 3 representation of $G_2(2)$ on 36 points [68].

10. Rank 3 representation of H S in 100 points [42,78,26].

11. Rank 3 representation of H J in 100 points [30,77,78].

12. Rank 3 representation of $PSU_4(3)$ on 162 points [78].

13. Rank 3 representationof McL on 275 points. A simple description
    may be given using graph 6. Given a point of $S(4,7,23)$ the
    253 blocks fall into two classes $\aleph_1 \aleph_2$, the blocks through the
    point (77) and the others (176). Now switch graph 6 with respect
    to $(\aleph_1\aleph_2)$ take $\bar{\aleph}_1$, and add the 22 points of $S(4,7,23)$ with
    the following adjacencies:a point is adjacent to each non incident

block of $\aleph_1$, and to every incident block of $\aleph_2$(see Conway [17] and also [55,78]).

14. Rank 3 representation of $G_2(4)$ on 416 vertices [78].

15. Rank 3 representation of Suz on 1782 vertices [78,79,81]

16. Rank 3 representation of Fi_{22} on 3510 vertices [78].

17. Rank 3 representation of Fi_{23} on 31671 vertices [78].

18. Rank 3 representation of Fi_{24} on 306.936 vertices [78].

19. Rank 3 representation of $2^{11}.M_{24}$ on 2048 vertices. This is related to Golay code; see [26].

20. Rank 3 representation of M_{24} over $2.M_{12}$. The 1288 vertices of the graph are the 1288 partitions of the 24 points of $S(5,8,24)$ in two subsets of length 12 on which M_{12} acts in non equivalent ways. Two vertices are adjacent if the 2 pairs of dodecads intersect in $(4,8,4,8)$ points.

21. Rank 3 representation of Co.2 over $2.PSU_6(2)$ In the Leech lattice take two points at distance $4\sqrt{2}$ and the two spheres, centered at these points, of some radius. Co.2 alts on the 2300 pairs of opposite points of the lattice which belong to the intersection sphere (see [17]).

22. Rank 3 representation of $PSU_6(2)$ over $PSU_4(3)$. Again in the Leech lattice, take a triangle of type 222; the stablizer of the vertices of the triangle acts on the 408 points which complete the triangle in a tetrahedron of type 222222.

23. Rank 3 representation of Rudvallis group on $'F_4(2)$. [18]

9. Combinatorial strongly regular graphs

The combinatorial point of view leads also to some interesting classes of s.r.graphs. Moreover some graphs are characterized only by combinatorial relations. We have already mentioned the result of Seidel [66] about graphs having -3 as smallest eigenvalue. In his proof he uses the previous results of Chang [14] (also a result of Connor [16]), of Hoffman [46] concerning the triangular association schemes $T(m)$ and the result of Shrikande [69] on the L_2-association schemes $L_2(n)$. In fact the parameters determines the two classes of graphs with 2 exceptions. For $v = 28 \ k = 12 \ \lambda = 6$, beside $T(8)$ there exist 3 other non isomorphic graphs [15] obtained from $T(8)$ by the switching process [65]. For $v = 16 \ k = 6 \ \lambda = 2$ there exists a non geometric graph with those parameters. In the same direction Bussemaker and Seidel [11] have proved the existence of more than 80 non-isomorphic graphs $L_2(6)$, and more than 23 graphs of type $NL_2(6)$.

Other classes of s.r. graphs have been constructed by various means.

Mesner [56]using a result of Ray-Chaudhuri [60] constructed two classes (in fact only one) of s.r graphs of negative latin square type (i.e. having the some parameters as $L_{-g}(n)$).

Bose and Shrikande [10] have constructed a large member of s,r graphs with $\lambda = \mu$ of type $L_2(2r)$, $NL_2(2r)$ and with parameters $(vk\lambda) = (4r^2-1, 2r^2, r^2)$. Also Wallis [83,84] has studied graphs with $\lambda = \mu$ and constructed other types of graphs using affine resolvable designs; he also showed the existence of at least 2 non-isomorphic graphs with parameters $(n^2(n+2), n(n+1),n)$ for $n = p^{\alpha}$.

These graphs where first constructed by Hall [29] when $p = 2$ and Ahrens and Szekeres [1] in the general case. The construction of Ahrens and Szekeres yields another class of s.r. graphs (K3). If $p \neq 2$ this construction is closely related to the construction of a graph on an hermitian parabola in an affine plane.

Other constructions by means of Hadamard matrices may be found in Hall [28] and in Goethals and Seidel [26]. In the last paper some other interesting graphs, related to quasi symetric designs (designs which have only two types of intersection) are also constructed. In particular there is a very neat study of the $S(5,8,24)$ an its derived graphs. Let us mention that the conjecture about the existence of a s.r graph with $v = 1288$ [26, p. 613] is true (see rank 3 towers).

K Combinatorial rank 3 graphs

1. Lattice square graphs $L_g(n)$. Given $g - 2$ orthogonal lattice squares of order n, the graph is constructed on the n^2 cells of the square. Two vertices are adjacent if the cells are in the same row, of column, of if they contain the same letter.

2. Triangular graphs. They correspond to a rank 3 representation of $Sym(n)$ acting on the pairs. Two vectices are adjacent if the pairs contain a commun index (see also c.1).

3. Graph of a partial geometry of type $(\lambda+2, \lambda, 1)$ $\lambda = p^\alpha$ see Ahrens and Szekeres [1].

4. Line graph of the same geometry.

5. Negative lattice square graphs (see [56,10]).

6. Line graph of a $S(2,\tilde{k},\tilde{v})$; adjacent vertices = intersecting blocks.

S.l.	v	k	ℓ	λ	μ	p_1	p_2
S.l.	56	10	45	0	2	7	-5
2.	77	16	60	0	4	11	-5
3.	50	7	42	0	1	5	-5
4.	105	32	72	4	12	19	-5
5.	120	42	77	8	18	23	-5
6.	253	112	140	36	60	51	-5
7.	176	70	105	18	34	35	-5
8.	175	72	102	20	36	35	-5
9.	36	14	21	4	6	7	-5
10.	100	22	77	0	6	15	-5
11.	100	36	63	14	12	7	-13
12.	162	56	105	10	24	31	-5
13.	275	112	162	30	56	55	-5
14.	416	100	315	36	20	7	-41
15.	1782	416	1365	100	96	31	-41
16.	3510	693	2816	180	126	17	-127
17.	31671	3510	28160	693	351	17	-703
18.	306936	31671	275264	3510	3240	161	-703
19.	2048	759	1288	310	264	17	-111
20.	1288	495	792	206	180	17	-71

	v	k	λ	ℓ	μ	ρ_1	ρ_2
21.	2300	891	378	1408	324	17	-127
22.	1408	567	246	840	216	17	-79
23.	4060	1755	730	2304	780	129	-31

	v	k	λ	ℓ	μ	$(\rho_1,\ \rho_2)$	
K 1.	n^2	$g(n-1)$	$(n-g+1)(n-1)$	$(g-1)(g-2)+n-2$	$g(g-1)$	$2g-1,$	$2g-2n-1$
2.	$\binom{n}{2}$	$2(n-2)$	$n-2$	$\binom{n-2}{2}$	4	$2n-7,$	-3
3.	q^3	$(q-1)(q+2)$	$q-2$	$(q-1)^2(q+1)$	$q+2$	$2q+3,$	$3-2q$
4.	$q^2(q+2)$	$q(q+1)$	q	$(q+1)^2(q-1)$	q	$2q-1,$	$-2q-1$
5.	n^2	$g(n+1)$	$(g+1)(g+2)-n-2$	$(n-g-1)(n+1)$	$g(g+1)$	$-2g+1,$	$-2g+2n-1$
6.	$\dfrac{\widetilde{v}(\widetilde{v}-1)}{\widetilde{k}(\widetilde{k}-1)}$	$\widetilde{k}\,\dfrac{\widetilde{v}-\widetilde{k}}{\widetilde{k}-1}$	$(\widetilde{k}-1)^2\,\dfrac{\widetilde{v}-1}{\widetilde{k}-1}-2$	$\dfrac{(\widetilde{v}-\widetilde{k})(\widetilde{v}-\widetilde{k}^2+\widetilde{k}-1)}{\widetilde{k}(\widetilde{k}-1)}$	$\widetilde{k}^2$	$2\widetilde{k}-1,$	$2\widetilde{k}-1-2\,\dfrac{\widetilde{v}-\widetilde{k}}{\widetilde{k}-1}$

	v	k	ℓ
C.1.	$\dfrac{(q^{n+1}-1)(q^n-1)}{(q+1)(q-1)^2}$	$\dfrac{q(q+1)(q^{n-1}-1)}{q-1}$	$\dfrac{q^4(q^{n-1}-1)(q^{n-2}-1)}{(q+1)(q-1)^2}$
2.3	$\dfrac{q^{2n}-1}{q-1}$	$q\,\dfrac{q^{2n-2}-1}{q-1}$	q^{2n-1}
$4^{\pm}$	$\dfrac{(q^n\mp 1)(q^{n-1}\pm 1)}{q-1}$	$\dfrac{q(q^{n-1}\mp 1)(q^{n-2}\pm 1)}{q-1}$	q^{2n-2}
$5^{(*)}$	$\dfrac{(q^{n-1}\mp 1)(q^n\pm 1)}{q^2-1}$	$\dfrac{q^2(q^{n-3}\mp 1)(q^{n-2}\pm 1)}{q^2-1}$	q^{2n-3}
6.	$\dfrac{(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)}$	$\dfrac{q(q^8-1)(q^3+1)}{q-1}$	$\dfrac{q^8(q^5-1)(q^4+1)}{q-1}$
7.	$\dfrac{(q^8-1)(q^3+1)}{q-1}$	$\dfrac{q(q^5-1)(q^2+1)}{q-1}$	$\dfrac{q^6(q^5-1)}{q-1}$
8.	$(q^5+1)(q^3+1)$	$q^3(q^2+1)$	q^8
$9^{\pm}$	$2^{2n-1}\pm 2^{n-1}$	$2^{2n-2}-1$	$2^{2n-2}\pm 2^{n-1}$
10.	$\dfrac{3^n(3^n\pm 1)}{2}$	$\dfrac{3^{n-1}(3^n\mp 1)}{2}$	$3^{2n-1}\pm 3^{n-1}\cdot 2 -1$
11.	q^{2n-2}	$(q+1)(q^{n-1}-1)$	$q(q^{n-1}-1)(q^{n-2}-1)$
$12^{\pm}$	q^{2n}	$(q^n\mp 1)(q^{n-1}\pm 1)$	$q^{n-1}(q-1)(q^{n+1}\mp 1)$
13.	$4\alpha+1=q$	2α	2α
$14^{(*)}$	$\dfrac{q^{n-1}(q^n\pm 1)}{q+1}$	$(q^{n-1}\mp 1)(q^{n-2}\pm 1)$	$\dfrac{q^{n-2}(q^{n-1}\mp 1)(q^2-q-1)}{q+1}$

(*) upper sign if n is odd, lower sign if n is even

	λ	μ	$\{\rho_1, \rho_2\}$
C.1.	$\dfrac{q^n-1}{q-1} + q^2 - 2$	$(q+1)^2$	$2q+1,\ -2\dfrac{q^{n-1}-1}{q-1} + 2q + 3$
2.3.	$\dfrac{q^{2n-2}-1}{q-1} - 2$	$\dfrac{q^{2n-2}-1}{q-1}$	$2q^{n-1}+1,\ -2q^{n-1}+1$
4.$\pm$	$\dfrac{q(q^{n-2}\mp 1)(q^{n-2}+q)}{q-1} + q - 1$	$\dfrac{(q^{n-1}\mp 1)(q^{n-2}+1)}{q-1}$	$\pm 2q^{n-2}+1,\ \mp 2q^{n-1}+1$
5.	$\dfrac{q^3(q^{n-4}+1)(q^{n-4}\mp q)}{q^2-1} + q^2 - 1$	$\dfrac{(q^{n-3}\mp 1)(q^{n-2}+1)}{q^2-1}$	$\pm 2q^{n-2}+1,\ \mp 2q^{n-3}+1$
6.	$\dfrac{q^2(q^2+1)(q^5-1)}{q-1} + q - 1$	$\dfrac{(q^4-1)(q^3+1)}{q-1}$	$2q^3+1,\ -2\dfrac{q^4(q^5-1)}{q-1} + 1$
7.	$\dfrac{q^2(q^3-1)(q+1)}{q-1} + q - 1$	$\dfrac{(q^3-1)(q^2+1)}{q-1}$	$2q^2+1,\ \dfrac{-2q^3(q^3-1)}{q-1} + 1$
8.	q^3-1	q^2+1	$2q^2+1,\ -2q^3+1$
9.$\pm$	$2^{2n-3}-2$	$2^{2n-3}\mp 2^{n-2}$	$1\pm 2^{n-1},\ -1\mp 2^n$
10.	$\dfrac{3^{n-1}(3^{n-1}\mp 1)}{2}$	$\dfrac{3^{n-1}(3^{n-1}\mp 1)}{2}$	$3^{n-1}\cdot 2-1,\ -3^{n-1}\cdot 2-1$
11.	$q^{n-1}+q^2-q-2$	$q(q+1)$	$2q+1,\ -2q^{n-1} + 2q+1$
12.$\pm$	$q(q^{n-1}\mp 1)(q^{n-2}\pm 1)+q-2$	$q^{n-1}(q^{n-1}\pm 1)$	$\pm 2q^{n-1}+1,\ \pm 2q^{n-1}\mp 2q^n+1$
13.	$\alpha - 1$	α	$\sqrt{q},\ -\sqrt{q}$
14.(*)	$q^{2n-5}(q+1)\pm q^{n-2}(q-1)-q$	$q^{n-3}(q+1)(q^{n-2}\pm 1)$	$\pm q^{n-2}+1,\ \mp 2q^{n-3}(q^2-q-1)+1$

(*) upper sign if n is odd, lower sign if n is even

	v	k	ℓ
15.	$\dfrac{q^{n-1}(q^n-1)}{q-1}$	$\dfrac{q^2(q^{n-1}-1)}{q-1}$	$\dfrac{q(q^n-1)(q^{n-2}-1)}{q-1} + q-1$
16.	q^6	$(q+1)(q^3-1)$	$q(q^2-1)(q^3-1)$
17.	2^{3r}	$2^{2r}+2^r-2$	$(2^{2r}-1)(2^r-1)$
18.	$2^{2r}(2^r+2)$	$2^r(2^r+1)$	$(2^{2r}-1)(2^r+1)$
19.	q^2	$m(q-1)$	$(q+1-m)(q-1)$
20.	q^3	$\dfrac{(q-1)(q^2+1)}{2}$	$\dfrac{(q+1)(q^2-1)}{2}$

27

λ	μ	$\{\rho_1, \rho_2\}$
15. $\dfrac{q^n-1}{q-1} + q^2 - 2q - 1$	q^2	$2q-1,\ -2\dfrac{q^n-1}{q-1} + 2q + 1$
16. $q^3 + q^2 - q - 2$	$q(q+1)$	$2q+1,\ -2q^3 + 2q + 1$
17. $2^r - 2$	$2^r + 2$	$2^{r-1}+3,\ -2^{r-1}+3$
18. 2^r	2^r	$2^{r-1}-1,\ -2^{r-1}-1$
19. $(m-1)(m-2)+q-2$	$m(m-1)$	$2m-1,\ 2m-2q-1$
20. $\dfrac{(q-1)^3}{4} - 1$	$\dfrac{(q-1)(q^2+1)}{4}$	$q^2,\ -q$

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